

A consistency condition for the vector potential in multiply-connected domains

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Abstract

A classical problem in electromagnetics concerns the representation of the electric and magnetic fields in the low-frequency or static regime, where topology plays a fundamental role. For multiply connected conductors, at zero frequency the standard boundary conditions on the tangential components of the magnetic field do not uniquely determine the vector potential. We describe a (gauge-invariant) consistency condition that overcomes this non-uniqueness and resolves a longstanding difficulty in inverting the magnetic field integral equation.

1 Introduction

We consider the problem of exterior electromagnetic scattering in the frequency domain, with particular attention paid to the behavior of the electric and magnetic fields in the static limit. Following standard practice, we write $\mathbf{E}^{\text{tot}}(\mathbf{x}) = \mathbf{E}^{\text{in}}(\mathbf{x}) + \mathbf{E}(\mathbf{x})$ and $\mathbf{H}^{\text{tot}}(\mathbf{x}) = \mathbf{H}^{\text{in}}(\mathbf{x}) + \mathbf{H}(\mathbf{x})$, where $\{\mathbf{E}^{\text{in}}, \mathbf{H}^{\text{in}}\}$ describe known incident electric and magnetic fields, $\{\mathbf{E}, \mathbf{H}\}$ denote the scattered field of interest, and $\{\mathbf{E}^{\text{tot}}, \mathbf{H}^{\text{tot}}\}$ denote the *total* fields. We write Maxwell’s equations in the form

$$\begin{aligned}\nabla \times \mathbf{H}^{\text{tot}} &= -i\omega\epsilon\mathbf{E}^{\text{tot}} \\ \nabla \times \mathbf{E}^{\text{tot}} &= i\omega\mu\mathbf{H}^{\text{tot}}.\end{aligned}\tag{1.1}$$

where ϵ, μ are the permittivity and permeability, and we define the wavenumber by $k = \omega\sqrt{\epsilon\mu}$. The scattered field is assumed to satisfy the Sommerfeld-Silver-Müller radiation condition:

$$\lim_{r \rightarrow \infty} \left(\mathbf{H} \times \frac{\mathbf{r}}{r} - \frac{\mu}{\epsilon} \mathbf{E} \right) = o\left(\frac{1}{r}\right).\tag{1.2}$$

For a perfect conductor Ω , two homogeneous conditions to be enforced on Γ , the boundary of Ω , are [1, 2]:

$$\mathbf{n} \times \mathbf{E}^{\text{tot}} = 0, \quad \mathbf{n} \cdot \mathbf{H}^{\text{tot}} = 0.\tag{1.3}$$

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It is also well-known that, on Γ , we must have

$$\mathbf{n} \times \mathbf{H}^{\text{tot}} = \mathbf{J}, \quad \mathbf{n} \cdot \mathbf{E}^{\text{tot}} = \rho/\epsilon, \quad (1.4)$$

assuming that the scattered fields are induced by a physical surface current \mathbf{J} and a corresponding charge ρ with

$$\nabla_\Gamma \cdot \mathbf{J} = i\omega\rho, \quad (1.5)$$

where ∇_Γ denotes the surface divergence.

Our primary interest here is with the classical representation of electromagnetic fields in terms of the vector and scalar potential (in the Lorenz gauge):

$$\mathbf{E} = i\omega\mathbf{A} - \nabla\phi \quad (1.6)$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \quad (1.7)$$

where $\mathbf{A}[\mathbf{J}](\mathbf{x}) = \mu \int_\Gamma g_k(\mathbf{x} - \mathbf{y}) \mathbf{J}(\mathbf{y}) dA_{\mathbf{y}}$ and $\phi(\mathbf{x}) = \frac{1}{i\omega\epsilon\mu} \nabla \cdot \mathbf{A}$, with $g_k(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}$. In scattering problems, \mathbf{J} , the unknown surface current, is a tangential vector field.

Using the first condition in (1.4), with care in taking the limit as \mathbf{x} approaches the boundary Γ from the exterior domain, we obtain the magnetic field integral equation (MFIE):

$$\frac{1}{2} \mathbf{J}(\mathbf{x}) - \mathbf{M}_k[\mathbf{J}](\mathbf{x}) = \mathbf{n}(\mathbf{x}) \times \mathbf{H}^{\text{in}}(\mathbf{x}) \quad (\mathbf{x} \in \Gamma), \quad (1.8)$$

where

$$\mathbf{M}_k[\mathbf{J}](\mathbf{x}) = \mathbf{n}(\mathbf{x}) \times \nabla \times \int_\Gamma g_k(\mathbf{x} - \mathbf{y}) \mathbf{J}(\mathbf{y}) dA_{\mathbf{y}}. \quad (1.9)$$

It is a second-kind Fredholm equation, originally suggested by Maue [3]. See [4, 5, 6, 7] for detailed discussions.

While the MFIE has a sequence of spurious resonances at higher and higher frequencies, below the first such resonance it yields an invertible and well-conditioned linear system, so long as the scatterer is simply-connected. When the scatterer is topologically non-trivial, however, the MFIE breaks down, having a non-trivial null-space at zero frequency. In the static limit, the dimension of the null-space is equal to the genus of the surface. While this problem has been carefully analyzed [8, 4, 9], no simple remedy has been provided to date.

In this paper, we show that the difficulties encountered by the MFIE can be resolved through the enforcement of an apparently new consistency condition on the vector potential, involving line integrals around *B-cycles* (see Fig. 1). Like the Aharonov-Bohm effect, it illustrates the fundamental role of the vector potential, but in a classical regime. We begin with the static case, since it is mathematically simpler and of importance in its own right.

2 Magnetostatics

In the zero frequency limit, Maxwell's equations are generally said to uncouple, with *scattered* electrostatic and magnetostatic fields denoted by $(\mathbf{E}_0, \mathbf{H}_0)$, respectively. In the unbounded domain $\mathbb{R}^3 \setminus \overline{\Omega}$ exterior to Ω , \mathbf{E}_0 satisfies

$$\nabla \times \mathbf{E}_0 = 0, \quad \nabla \cdot \mathbf{E}_0 = 0, \quad (2.1)$$

subject to the boundary condition

$$\mathbf{n} \times \mathbf{E}_0 = -\mathbf{n} \times \mathbf{E}_0^{\text{in}}|_\Gamma, \quad (2.2)$$

where \mathbf{n} denotes the outward normal to Ω . In $\mathbb{R}^3 \setminus \overline{\Omega}$, \mathbf{H}_0 satisfies

$$\nabla \times \mathbf{H}_0 = 0, \quad \nabla \cdot \mathbf{H}_0 = 0, \quad (2.3)$$

subject to the boundary condition

$$\mathbf{n} \cdot \mathbf{H}_0 = -\mathbf{n} \cdot \mathbf{H}_0^{\text{in}}|_\Gamma. \quad (2.4)$$

Spanning surfaces S_2 and S_1

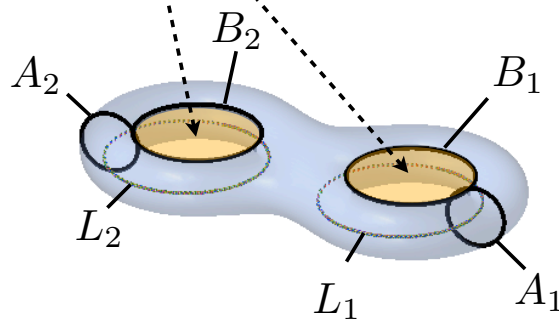


Figure 1: A surface of genus 2. The A -cycles are loops whose spanning surface is in the interior of the scatterer. The B -cycles have spanning surfaces S_1, S_2 which lie in the exterior domain. The loops L_1, L_2 lie in the interior domain and pass through the A -cycles.

Vector fields that are both curl-free and divergence-free are called harmonic vector fields.

A fundamental difficulty arises in the static theory that is topological in nature. More precisely, if the number of connected components of Ω is denoted by m , then there is an m -dimensional space of nontrivial vector fields \mathbf{E}_0^h in $\mathbb{R}^3 \setminus \overline{\Omega}$, satisfying

$$\nabla \times \mathbf{E}_0^h = 0, \quad \nabla \cdot \mathbf{E}_0^h = 0, \quad \mathbf{n} \times \mathbf{E}_0^h = 0|_{\Gamma}. \quad (2.5)$$

They are generally referred to as (exterior) Dirichlet fields and correspond to setting the electrostatic potential ϕ on each disjoint conductor to a different constant. This space of fields is, of course, essential in studying capacitance problems in a system of disjoint conductors, where $\mathbf{E}_0^h = -\nabla\phi$.

Here, we concentrate on magnetostatics, where nontrivial solutions $\mathbf{H}_0(\mathbf{x})$ to

$$\nabla \times \mathbf{H}_0 = 0, \quad \nabla \cdot \mathbf{H}_0 = 0, \quad \mathbf{n} \cdot \mathbf{H}_0 = 0|_{\Gamma}, \quad (2.6)$$

are referred to as interior or exterior Neumann fields, depending on whether $\mathbf{x} \in \Omega$ or $\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}$, respectively. We let H^+ denote the space of exterior Neumann fields and H^- denote the space of interior Neumann fields. It is a classical fact that the dimension of the spaces H^+ and H^- is g , where g is the genus of the boundary Γ [4].

In short, the boundary condition (2.4) alone determines a unique field only in the simply connected case ($g = 0$). A natural representation in that setting is to seek \mathbf{H}_0 as the gradient of the magnetic scalar potential, $\mathbf{H}_0 = \nabla\Psi$, with Ψ a single-valued harmonic function satisfying

$$\mathbf{n} \cdot \nabla\Psi = -\mathbf{n} \cdot \mathbf{H}_0^{\text{in}}. \quad (2.7)$$

In the multiply connected case, additional data is needed to make the magnetostatic problem well-posed, all of which are designed to account for current loops that may be flowing through the handles of the domain. One such condition is to require that the line integrals of \mathbf{H}_0 around each A -cycle be specified:

$$\int_{A_j} \mathbf{H}_0 \cdot d\mathbf{s} = \alpha_i. \quad (2.8)$$

It can be shown that the solution to (2.3,2.4,2.8) is unique (see, for example, [10, 11, 4, 12, 13, 14]).

The use of the scalar potential can be extended to the multiply connected case in two ways - either by introducing g cuts on the boundary Γ and allowing for a potential jump across each cut [16], or by introducing g current loops in the interior domain Ω that span the fundamental group of Ω (essentially one passing through each A -cycle - see Fig. 1) [10, 12]. In the latter case, it is straightforward to see that one

can represent \mathbf{H}_0 as

$$\mathbf{H}_0 = \nabla \Psi + \frac{1}{\mu} \nabla \times \sum_{i=1}^g \alpha_i \mathbf{A}[\mathcal{L}_i],$$

where \mathcal{L}_i is a loop of unit current density flowing through L_i (Fig. 1), allowing the scalar potential Ψ to be single-valued.

It is also possible to extend the scalar potential approach to the full Maxwell equations at non-zero frequencies through a generalization of the Lorenz-Debye-Mie formalism given in [17, 18], but this involves non-physical variables. Unlike the above formulation, the MFIE (1.8) uses a physical unknown, the surface current \mathbf{J} , and extends naturally away from the static limit through the representation (1.6)-(1.7). In this paper we seek to better understand, in the static case, the range and null-space of the MFIE.

3 Magnetostatics using the vector potential

There is a substantial literature on magnetostatics and the representation of harmonic vector fields in the form of a curl (see, for example, [8, 9, 12, 14, 15, 16]). We do not seek to review the theory here, except where it is of direct relevance to the MFIE. At zero frequency, the MFIE takes the form

$$\left[\frac{1}{2} I - M_0 \right] \mathbf{J} = \mathbf{n} \times \mathbf{H}^{\text{in}}, \quad (3.1)$$

where I denotes the identity operator and we have dropped the argument \mathbf{x} for the sake of clarity. We let $\{H_j^+ : j = 1, \dots, g\}$ denote a basis for the exterior Neumann fields and $\{H_j^- : j = 1, \dots, g\}$ denote a basis for the interior Neumann fields. We let $\{Z_j^+ = H_j^+|_\Gamma : j = 1, \dots, g\}$ denote the boundary values of the exterior Neumann fields and $\{Z_j^- = H_j^-|_\Gamma : j = 1, \dots, g\}$ denote the boundary values of interior Neumann fields, which are perforce vector fields tangent to Γ . As shown in [9, 4, 8], the relevant null-spaces are known to be

$$\begin{aligned} N\left(\frac{1}{2}I - M_0\right) &= \{\mathbf{n} \times Z_j^+ : j = 1, \dots, g\} \\ N\left(\frac{1}{2}I - M'_0\right) &= \{Z_j^- : j = 1, \dots, g\}, \end{aligned} \quad (3.2)$$

where M' is the adjoint of M . From Fredholm theory, the solvability conditions for the MFIE are, therefore, that the inner products $(\mathbf{n} \times \mathbf{H}^{\text{in}}, Z_j^-) = 0$ for $j = 1, \dots, g$. This is always the case for an incoming field that is generated by currents lying outside of a simply connected neighborhood of Ω [9]. In short, for the physical scattering problem, the operator $\frac{1}{2}I - M_0$ is *rank* deficient but not *range* deficient, and it remains only to develop a set of constraints that make it invertible.

We turn now to the main point of the present paper: that such constraints can be found and that they come from an analysis of the *electric* field in the limit $\omega \rightarrow 0$. We begin by noting that on a perfect conductor, the vanishing of the total tangential electric field (1.3) and Stokes' theorem allows us to write

$$\int_{B_j} \mathbf{E} \cdot d\mathbf{s} = - \int_{B_j} \mathbf{E}^{\text{in}} \cdot d\mathbf{s} = - \int_{S_j} i\omega\mu \mathbf{H}^{\text{in}} \cdot \mathbf{n} dA,$$

where S_j is a spanning surface for the B -cycle B_j . Dividing both sides by $i\omega$, using the representation (1.6), and noting that the line integral of a gradient vanishes, we have

Theorem 1. *Let Ω be a multiply-connected perfect conductor and let B_j be a B -cycle and S_j be a spanning surface for B_j . Then*

$$\int_{B_j} \mathbf{A} \cdot d\mathbf{s} = -\mu \int_{S_j} \mathbf{H}^{\text{in}} \cdot \mathbf{n} dA. \quad (3.3)$$

This condition makes sense for any $\omega \geq 0$. Alternatively, if we have access to the vector potential \mathbf{A}^{in} which defines the incoming field, the same analysis yields the even simpler condition:

$$\int_{B_j} \mathbf{A} \cdot d\mathbf{s} = - \int_{B_j} \mathbf{A}^{\text{in}} \cdot d\mathbf{s}. \quad (3.4)$$

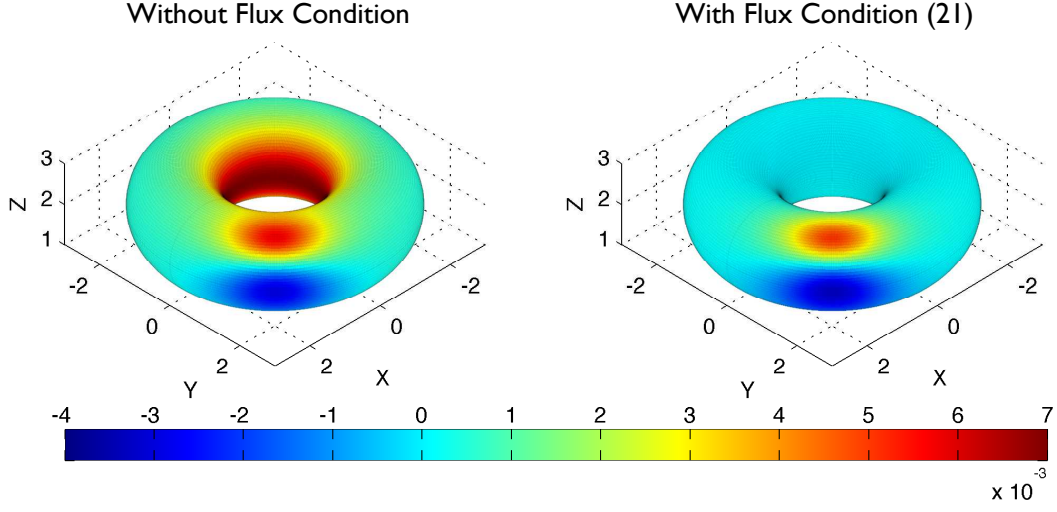


Figure 2: The induced surface current on a torus at wavenumber $k = 10^{-16}$ without (left) and with (right) the auxiliary consistency (flux) condition. The real part of the current in the azimuthal direction is shown. The incident field is due to a unit strength current loop of radius .5 located at (3, 3, 4).

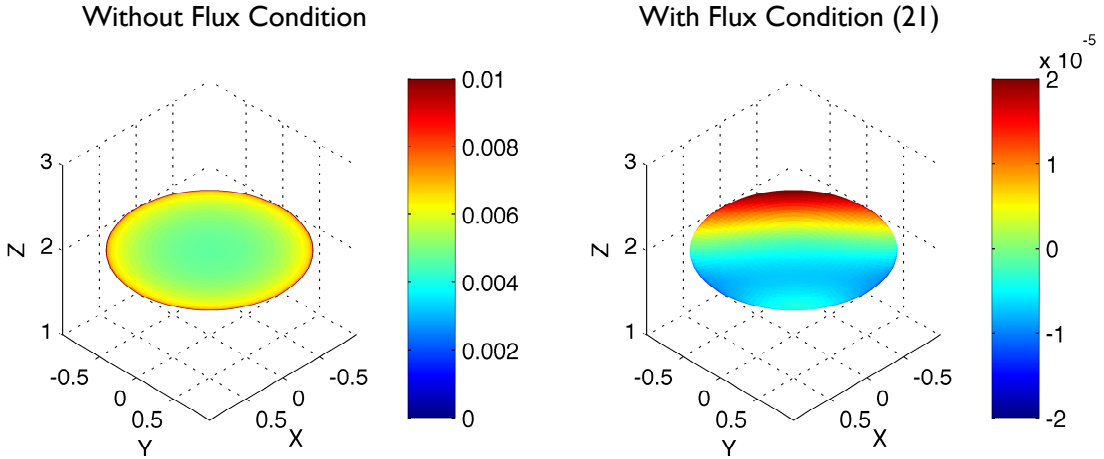


Figure 3: The real part of $\mathbf{H}^{\text{tot}} \cdot \hat{\mathbf{z}}$ on surfaces spanning the holes of the tori in Fig. 2. Note the different scales on the color bars, as well as the change in sign of the point-wise flux on the right. The net flux through the left surface is 0.019 while, on the right, it is zero to machine precision.

Theorem 2. *Let Ω be a multiply-connected perfect conductor of genus g with B -cycles B_1, \dots, B_g . Then the MFIE augmented by conditions of the form (3.3) or (3.4) for $j = 1, \dots, g$ has a unique solution.*

Proof. As noted above, the equation $(\frac{1}{2}I - M_0)\mathbf{J} = \mathbf{n} \times \mathbf{H}_0^{\text{in}}$ is solvable for any physically meaningful right hand side. Thus the only issue is that of uniqueness; we need to show that the combined null-space of the operator $(\frac{1}{2}I - M_0)$ and (3.3) is trivial. The null-space of the integral operator is spanned by $\{\mathbf{n} \times \mathbf{Z}_j^+ : j = 1, \dots, g\}$. Hence, if \mathbf{J} lies in the null-space of this integral operator, then the exterior field takes the form

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}[\mathbf{J}] = \sum_{j=1}^g \beta_j \mathbf{H}_j^+, \quad (3.5)$$

see [9]. Under the correspondence between vector fields and 2-forms

$$h_1 \partial_{x_1} + h_2 \partial_{x_2} + h_3 \partial_{x_3} \leftrightarrow h_1 dx_2 \wedge dx_3 + h_2 dx_3 \wedge dx_1 + h_3 dx_1 \wedge dx_2, \quad (3.6)$$

the fields $\{\mathbf{H}_j^+ : j = 1, \dots, g\}$ are a basis for the relative cohomology group $H_{\text{dR}}^2(\mathbb{R}^3 \setminus \overline{\Omega}, \Gamma)$. The functionals

$$\mathbf{H} \mapsto \int_{S_j} \mathbf{H} \cdot \mathbf{n} dA = \frac{1}{\mu} \int_{B_j} \mathbf{A}[\mathbf{J}] \cdot d\mathbf{s} \text{ for } j = 1, \dots, g \quad (3.7)$$

span the dual space to this vector space. Hence if these integrals all vanish, then the coefficients $\{\beta_j\}$ must all vanish as well. \square

Remark 1. *Let us suppose now that we have discretized the MFIE (1.8) with $2N$ unknowns (2 degrees of freedom for the surface current \mathbf{J} at each of N points), resulting in the linear system $\mathbf{A}\mathbf{j} = \mathbf{b}$ and the constraints (3.4) by $\mathbf{C}\mathbf{j} = \mathbf{f}$, where \mathbf{C} is a $g \times 2N$ matrix. It is straightforward to show that the system*

$$[\mathbf{A} + \mathbf{Q}\mathbf{C}]\mathbf{j} = \mathbf{b} + \mathbf{Q}\mathbf{f}$$

has the same solution as the constrained equation. It is invertible so long as the range of \mathbf{Q} , a $2N \times g$ matrix, has a full rank projection onto the null vectors of the adjoint \mathbf{A}' . In the low frequency regime, it is sufficient to use $\mathbf{Q} = \mathbf{C}'$.

4 Time harmonic electromagnetics using the vector potential

As soon as $\omega \neq 0$, the MFIE (1.8) is formally invertible and there is no need for the incorporation of consistency constraints. Because there is a nearby singular problem, however, the linear system is extremely ill-conditioned at low frequency. Without the additional constraints the *condition number* is $O(\omega^{-2})$. In that regime, the incorporation of the constraints improves the condition number considerably.

5 Numerical Results

For illustration, we consider the problem of scattering from a torus: a genus one surface of revolution (Fig. 2), driven by a known current loop in the exterior. We have implemented a solver for the MFIE with and without the consistency condition (with a detailed discussion of the method to be reported at a later date). One convenient measure of the error in the solution is the flux of \mathbf{H}^{tot} through the hole in the torus, which should vanish on a perfect conductor (see Fig. 3).

6 Conclusions

In this paper, we have derived a simple consistency condition (Theorem 1) for the vector potential in the context of scattering from perfect conductors, valid at any frequency. It is of physical interest for three reasons: 1) it describes an interesting correlation between the electric and magnetic field that persists at zero frequency, 2) it enforces uniqueness for the magnetic field integral equation (MFIE) in multiply connected

domains in the static limit, and 3) it improves the stability and robustness of the MFIE in the low frequency regime.

It is, perhaps, worth noting that the role of the vector potential in the consistency condition is, in some sense, dual to its role in the Aharonov-Bohm effect. In the latter case, it is the line integral of the vector potential around A -cycles that is critical. That integral measures the *solenoidal (poloidal)* current flow on the torus, which induces a zero electromagnetic field in the exterior. The line integral of \mathbf{A} around B -cycles, on the other hand, is sensitive to the *toroidal* current flow on the surface. On a perfect conductor, that induced current exactly cancels the flux of the incoming magnetic field through the “holes.”

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